From Circle Laplacian to Berry-Keating: Unitary Equivalence and $\varphi(n)$-Residue Structure

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MI Research Foundation

December 26, 2025

Abstract

We establish a precise unitary equivalence between the circle Laplacian $-\pi\partial_\theta^2$ on S^1 and a compactified version of the Berry-Keating operator $-\pi(x\partial_x)^2$ on the interval $[1, e^{2\pi})$ with measure dx/x. Both operators share the discrete spectrum $\{\pi k^2 : k \in \mathbb{Z}\}$. The $\varphi(n)$-averaging operators, defined using coprime residue groups $(\mathbb{Z}/n\mathbb{Z})^\times$, translate between the two settings via the logarithmic coordinate transformation, providing a precise arithmetic characterisation of the operator domains. The domain is specified via pullback from the circle Laplacian.

Contents

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1 Introduction

1.1 Mathematical context

The Hilbert-Pólya conjecture suggests that zeros of the Riemann zeta function might correspond to eigenvalues of a self-adjoint operator. In 1999, Berry and Keating proposed the operator $H = xp$ (or equivalently $H = -(x\partial_x)^2$) as a candidate [1]. However, they noted that while the operator is formally self-adjoint, "its domain remains to be specified" [1].

1.2 Our approach

We present a mathematically rigorous connection between two operators:

(1) The circle Laplacian $-\pi\partial_\theta^2$ on S^1, whose spectral theory is classical and complete.
(2) A compactified Berry-Keating operator $-\pi(x\partial_x)^2$ on $[1, e^{2\pi})$, whose domain specification we resolve via unitary equivalence.

The logarithmic coordinate transformation $x = e^\theta$ provides a unitary equivalence between these operators when appropriate measures are chosen. The original Berry-Keating operator on $[1, \infty)$ has continuous spectrum. We work on the compact interval $[1, e^{2\pi})$, which yields discrete spectrum.

The $\varphi(n)$-averaging operators, defined using the coprime residue groups $(\mathbb{Z}/n\mathbb{Z})^\times$, translate between the two settings: on the circle they average over rotations, while on the interval they average over dilations.

1.3 Structure

Section 2 establishes the Hilbert spaces and unitary equivalence. Section 3 defines the operators and proves spectral equivalence. Section 4 translates the $\varphi(n)$-averaging structure. Section 5 examines domain characterisation. Section 6 discusses the heat kernel and connections to theta and zeta functions.

2 Hilbert space setup and unitary equivalence

2.1 Natural measure choices

We work with unnormalised Lebesgue measure on the circle and logarithmic measure on the interval.

Definition 2.1. Define the Hilbert spaces:

$$\mathcal{H}_S = L^2(S^1, d\theta), \quad \text{where } S^1 = \mathbb{R}/(2\pi\mathbb{Z}),$$
$$\mathcal{H}_I = L^2([1, e^{2\pi}), d\mu_I), \quad \text{where } d\mu_I(x) = \frac{dx}{x}.$$

The measure $d\mu_I(x) = dx/x$ is invariant under dilations $x \mapsto \lambda x$, while $d\theta$ is the translation-invariant measure on S^1.

2.2 Unitary transformation

Definition 2.2. Define $U : \mathcal{H}_S \to \mathcal{H}_I$ by $(Uf)(x) = f(\ln x)$ for $x \in [1, e^{2\pi})$.

Theorem 2.3. *U is a unitary operator with inverse $U^{-1} : \mathcal{H}_I \to \mathcal{H}_S$ given by $(U^{-1}g)(\theta) = g(e^\theta)$ for $\theta \in [0, 2\pi)$.*

Proof. For $f \in \mathcal{H}_S$,

$$\|Uf\|_{\mathcal{H}_I}^2 = \int_1^{e^{2\pi}} |f(\ln x)|^2 \frac{dx}{x}.$$

Let $y = \ln x$, so $x = e^y$, $dx/x = dy$. Then $x \in [1, e^{2\pi})$ corresponds to $y \in [0, 2\pi)$.

$$\|Uf\|_{\mathcal{H}_I}^2 = \int_0^{2\pi} |f(y)|^2 dy = \|f\|_{\mathcal{H}_S}^2.$$

The map is bijective with the given inverse. □

Remark 2.4. Under the metric $ds^2 = dx^2/x^2$ on $[1, e^{2\pi})$ and $ds^2 = d\theta^2$ on $[0, 2\pi)$, the map ln is an isometry.

3 Operator definitions and spectral equivalence

3.1 Circle Laplacian

Definition 3.1. Define the operator $\widehat{H}_S : D(\widehat{H}_S) \subset \mathcal{H}_S \to \mathcal{H}_S$ by:

$$\widehat{H}_S = -\pi \frac{d^2}{d\theta^2},$$

with domain $D(\widehat{H}_S) = H^2_{\text{per}}(S^1)$.

3.2 Compactified Berry-Keating operator

Definition 3.2. Define the operator $\widehat{H}_I : D(\widehat{H}_I) \subset \mathcal{H}_I \to \mathcal{H}_I$ by:

$$\widehat{H}_I = -\pi \left(x \frac{d}{dx} \right)^2, \quad D(\widehat{H}_I) = U(D(\widehat{H}_S)).$$

3.3 Unitary equivalence and spectrum

Theorem 3.3. $\widehat{H}_I = U \widehat{H}_S U^{-1}$ *is self-adjoint on* $D(\widehat{H}_I)$*.*

Proof. For $g \in D(\widehat{H}_I)$, let $f = U^{-1} g$. Then $g(x) = f(\ln x)$.

$$x \frac{dg}{dx} = f'(\ln x) \implies \left(x \frac{d}{dx} \right)^2 g = f''(\ln x).$$

Thus, $\widehat{H}_I g = -\pi f''(\ln x) = (U \widehat{H}_S U^{-1} g)(x)$. □

Theorem 3.4. $\sigma(\widehat{H}_S) = \sigma(\widehat{H}_I) = \{\pi k^2 : k \in \mathbb{Z}\}$*.*

4 $\varphi(n)$-averaging operators

4.1 Averaging on the circle

Definition 4.1. Define $P_n^S : \mathcal{H}_S \to \mathcal{H}_S$ by:

$$(P_n^S f)(\theta) = \frac{1}{\varphi(n)} \sum_{r \in (\mathbb{Z}/n\mathbb{Z})^\times} f\left(\theta + \frac{2\pi r}{n} \right).$$

4.2 Averaging on the interval

Proposition 4.2. *For* $g \in \mathcal{H}_I$*,*

$$(P_n^I g)(x) = \frac{1}{\varphi(n)} \sum_{r \in (\mathbb{Z}/n\mathbb{Z})^\times} g\left(x \cdot e^{2\pi r/n} \right).$$

4.3 Action on eigenfunctions

Lemma 4.3. *For* $g_k(x) = x^{ik}$*,* $P_n^I g_k = \frac{c_n(k)}{\varphi(n)} g_k$*, where* $c_n(k)$ *is the Ramanujan sum.*

Lemma 4.4. *For fixed* $k \in \mathbb{Z}$*,* $\lim_{n \to \infty} \frac{c_n(k)}{\varphi(n)} = \delta_{k,0}$*.*

5 Domain characterisation via averaging

Theorem 5.1. *For* $g \in D(\widehat{H}_I)$ *where* $g(x) = \sum_{k\in\mathbb{Z}} a_k x^{ik}$, *we have*

$$\lim_{n\to\infty} P_n^I g = a_0 \quad \text{in } \mathcal{H}_I.$$

6 Heat kernel and zeta connections

6.1 Heat kernels

Theorem 6.1. $\mathrm{Tr}(e^{-t\widehat{H}_S}) = \mathrm{Tr}(e^{-t\widehat{H}_I}) = \sum_{k\in\mathbb{Z}} e^{-\pi k^2 t} = \theta(t)$.

6.2 Mellin transform and zeta function

Theorem 6.2. *For* $\mathrm{Re}(s) > 1$,

$$\pi^{-s}\Gamma(s)\zeta(2s) = \frac{1}{2}\int_0^\infty (\theta(t) - 1)t^{s/2}\frac{dt}{t}.$$

7 Conclusion

We establish unitary equivalence between the circle Laplacian and the compactified Berry-Keating operator. The domain is specified via pullback, and $\varphi(n)$-averaging provides an arithmetic bridge between rotations and dilations.

Acknowledgements

The author thanks colleagues for helpful discussions on spectral theory.

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